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THE MOTION OF A SATELLITE.

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(1). THE apparent orbit of a satellite around its primary will be in general an ellipse, which will be the projection of the real orbit of the satellite on a plane perpendicular to the line of sight, or to a right line drawn from the observer to the centre of the planet. Assuming that the orbit of the satellite around its primary is known, and that its motion is purely elliptical, or free of perturbation, we have the problem of determining from the known orbital motion of the satellite its apparent motion. The solution of this problem has been given by Bessel in the Astronomische Nachrichten, Bd. 9, p. 7; and again in his Astronomische Untersuchungen, Bd. 2, p. 37; and also by Marth in the Astronomische Nachrichten, Bd. 44, p. 113.

Denote by s and p the apparent distance and angle of position of a satellite, or its polar coordinates referred to the centre of the planet. The zero of the angle p is the circle of declination passing through the centre of the planet, this angle being counted from the north toward the east, and from 0° to 360°.

Let α and δ be the right ascension and declination of the planet, α' and δ' the same quantities for the satellite. From the spherical triangle between the pole of the equator, the planet and the satellite, we have

$$\left.\begin{aligned} \cos s &= \sin\delta\sin\delta' + \cos\delta\cos\delta'\cos(\alpha'-\alpha) \\ \sin s\cos p &= \cos\delta\sin\delta' - \sin\delta\cos\delta'\cos(\alpha'-\alpha) \\ \sin s\sin p &= \cos\delta'\sin(\alpha'-\alpha) \end{aligned}\right\}. \qquad (1)$$

If we denote by r the radius vector of the satellite, and by a and d its right ascension and declination seen from the planet, and also by ρ and ρ' the distances of the planet and the satellite from the Earth, we shall have by equating the values of the rectangular coordinates,

$$\begin{aligned}
\rho' \cos \delta' \cos \alpha' &= \rho \cos \delta \cos \alpha + r \cos d \cos a, \\
\rho' \cos \delta' \sin \alpha' &= \rho \cos \delta \sin \alpha + r \cos d \sin a, \\
\rho' \sin \delta' &= \rho \sin \delta \qquad\quad + r \sin d,
\end{aligned}$$

or by a simple transformation

$$\begin{aligned}
\rho' \cos \delta' \cos (\alpha' - \alpha) &= \rho \cos \delta + r \cos d \cos (a - \alpha), \\
\rho' \cos \delta' \sin (\alpha' - \alpha) &= \qquad\qquad\; r \cos d \sin (a - \alpha), \\
\rho' \sin \delta' &= \rho \sin \delta + r \sin d.
\end{aligned}$$

If we multiply equations (1) by ρ' we have by means of the last equations

$$\left.\begin{aligned}
\rho' \cos s &= \rho + r \,.\, [\sin d \sin \delta + \cos d \cos \delta \cos (a - \alpha)] \\
\rho' \sin s \cos p &= \quad\;\; r \,.\, [\sin d \cos \delta - \cos d \sin \delta \cos (a - \alpha)] \\
\rho' \sin s \sin p &= \quad\;\; r \,.\, \cos d \sin (a - \alpha)
\end{aligned}\right\} . \qquad (2)$$

If we place now

$$\xi = \frac{r}{\rho} \,.\, \cos d \sin (a - \alpha),$$

$$\eta = \frac{r}{\rho} \,.\, [\sin d \cos \delta - \cos d \sin \delta \cos (a - \alpha)],$$

$$\zeta = \frac{r}{\rho} \,.\, [\sin d \sin \delta + \cos d \cos \delta \cos (a - \alpha)],$$

equations (2) become

$$\left.\begin{aligned}
\rho' \cos s &= \rho . (1 + \zeta) \\
\rho' \sin s \cos p &= \rho . \eta \\
\rho' \sin s \sin p &= \rho . \xi
\end{aligned}\right\} . \qquad (3)$$

We have therefore

$$\operatorname{tang} s \sin p = \frac{\xi}{1 + \zeta},$$

$$\operatorname{tang} s \cos p = \frac{\eta}{1 + \zeta}.$$

If we assume $\operatorname{tang} s = s$, which may generally be done without sensible error, and put $x = s \sin p$, $y = s \cos p$; we shall have in seconds of arc

$$x = \frac{\omega \,.\, \xi}{1 + \zeta}: \qquad\qquad y = \frac{\omega \,.\, \eta}{1 + \zeta}:$$

where $\omega = 206264.8$. The equations (3) give the values of ρ', s and p when those of ξ, η and ζ are known. It is evident also that ζ has no influence on the value of p, and since it is generally very small with respect to unity its influence on the value of s is slight.

We have now to express the quantities ξ, η, ζ, in terms of the elements of the orbit of the satellite. Continuing the use of Bessel's notation, let N be the longitude of the ascending node of the orbit of the satellite on the Equator, and J its inclination to the Equator. Let u be the argument of the latitude, or the angle between the node and the satellite counted in the direction of the motion. From the right angled spherical triangle having u for a hypotenuse, and the sides d, and $a—N$, we have

$$\begin{aligned}\cos d \sin (a—N) &= \sin u \cos J,\\ \cos d \cos (a—N) &= \cos u,\\ \sin d &= \sin u \sin J.\end{aligned}$$

Eliminating N from the left hand side of these equations we have

$$\begin{aligned}\cos d \cos (a—\alpha) &= \sin u \sin (\alpha—N) \cos J + \cos u \cos (\alpha—N),\\ \cos d \sin (a—\alpha) &= \sin u \cos (\alpha—N) \cos J — \cos u \sin (\alpha—N),\\ \sin d &= \sin u \sin J.\end{aligned}$$

Denote the mean distance of the planet from the sun by ρ_0, and the semi-major axis of ths orbit of the satellite by $\rho_0 \sin \varDelta$; if we express the radius vector r by means of the semi-major axis the value of r will be

$$\rho_0 \sin \varDelta \, . \, r,$$

or approximately

$$\rho_0 \, . \, \frac{\varDelta}{\omega} \, . \, r.$$

The values of ξ, η, ζ, are therefore

$$\left.\begin{aligned}\omega.\xi &= r \, . \frac{\rho_0}{\rho} \varDelta . \left\{ \sin u \cos (\alpha—N) \cos J — \cos u \sin (\alpha—N) \right\}\\ \omega.\eta &= r \, . \frac{\rho_0}{\rho} \varDelta . \left\{ \sin u \left[\cos \delta \sin J — \sin \delta \cos J \sin (\alpha—N)\right] \right.\\ &\qquad\qquad \left. — \cos u \sin \delta \cos (\alpha—N) \right\}\\ \omega.\zeta &= r \, . \frac{\rho_0}{\rho} \varDelta . \left\{ \sin u \left[\sin \delta \sin J + \cos \delta \cos J \sin (\alpha—N)\right] \right.\\ &\qquad\qquad \left. + \cos u \cos \delta \cos (\alpha—N) \right\}\end{aligned}\right\} . \tag{4}$$

Introducing the auxiliary quantities f, F, g, G, h, H, by the formulæ

$$\begin{aligned}\sin f \cos F &= \cos (\alpha—N) \cos J,\\ \sin f \sin F &= —\sin (\alpha—N),\\ \cos f &= —\cos (\alpha—N) \sin J,\\ \sin g \cos G &= \cos \delta \sin J — \sin \delta \cos J \sin (\alpha—N),\\ \sin g \sin G &= —\sin \delta \cos (\alpha—N),\\ \cos g &= \cos \delta \cos J + \sin \delta \sin J \sin (\alpha—N),\\ \sin h \cos H &= \sin \delta \sin J + \cos \delta \cos J \sin (\alpha—N),\\ \sin h \sin H &= \cos \delta \cos (\alpha—N),\\ \cos h &= \sin \delta \cos J — \cos \delta \sin J \sin (\alpha—N),\end{aligned}$$

we have

$$\left.\begin{aligned} \omega.\xi &= r.\frac{\rho_0}{\rho}\Delta\sin f\sin(F+u) \\ \omega.\eta &= r.\frac{\rho_0}{\rho}\Delta\sin g\sin(G+u) \\ \omega.\zeta &= r.\frac{\rho_0}{\rho}\Delta\sin h\sin(H+u) \end{aligned}\right\}. \tag{5}$$

Denoting the mean longitude at the instant from which the time t is counted by E, the daily motion by λ, the longitude of the perihelion by P, the eccentricity of the orbit by e, the mean, the eccentric and the true anomalies by μ, ε and φ, we have

$$\left.\begin{aligned} \mu &= E+\lambda t-P = \varepsilon-e\sin\varepsilon \\ \operatorname{tang}\tfrac{1}{2}\varphi &= \sqrt{\frac{1+e}{1-e}}\,.\operatorname{tang}\tfrac{1}{2}\varepsilon \\ r &= 1-e\cos\varepsilon. \end{aligned}\right\}. \tag{6}$$

The preceding formulæ are those given by Bessel. The six auxiliary quantities f, F, g, G, h, H, are similar to those introduced by Gauss for computing the place of a planet (Zach's Monatliche Correspondenz, Mai, 1804, or Werke, Bd. 6, p. 94), and their geometrical meaning is readily seen. Thus, if we take the origin of rectangular coordinates at the centre of the planet, the line drawn from the planet to the Earth as the axis of x and take the axis of y in the plane of the declination circle, we shall find that f, g, h, are the arcs drawn from the north pole of the orbit of the satellite to the points where the axes of x, y and z cut the sphere; and that F, G, H, are the angles at this pole between the arcs drawn to the north pole of the Equator, and those drawn to the points where the sphere is cut by the axes of x, y and z.

By introducing a single additional auxiliary quantity Marth has given a very simple and elegant method of computing s and p. Thus if we compute the angle k by the formulæ

$$\left.\begin{aligned} \sin h\sin k &= \cos(\alpha-N)\sin J \\ \sin h\cos k &= -\sin(\alpha-N)\sin J\sin\delta-\cos J\cos\delta \end{aligned}\right\}, \tag{7}$$

and denote by σ the supplement of the angle between the line drawn from the Earth to the planet and the radius vector of the satellite, we have from the spherical triangle formed by the directions of the selines and the pole of the satellite's orbit,

$$\left.\begin{aligned} \sin\sigma\sin(p-k) &= \cos(H+u) \\ \sin\sigma\cos(p-k) &= \sin(H+u)\cos h \\ \cos\sigma &= \sin(H+u)\sin h \end{aligned}\right\}. \tag{8}$$

We evidently have

$$\rho' \sin s = r \sin \sigma,$$
$$\rho' \cos s = r \cos \sigma + \rho,$$

and therefore we can find ρ', s and p from the equations

$$\left.\begin{aligned} \rho' \sin s \sin (p-k) &= r \cos (H+u) \\ \rho' \sin s \cos (p-k) &= r \sin (H+u) \cos h \\ \rho' \cos s &= r \sin (H+u) \sin h + \rho \end{aligned}\right\}. \qquad (9)$$

By putting

$$\sin (H+u) \sin h + \frac{\rho}{r} = \frac{1}{\tau}$$

and eliminating ρ' we have for computing s and p,

$$\operatorname{tang} s \sin (p-k) = \tau \cos (H+u)$$
$$\operatorname{tang} s \cos (p-k) = \tau \sin (H+u) \cos h.$$

If we wish to use the angle σ we may compute s by means of the equations

$$\sin s = \frac{r}{\rho} . \sin (\sigma - s),$$

and for ρ' we have

$$\rho' = \rho . \frac{\sin \sigma}{\sin (\sigma - s)}.$$

Both these equations come directly from the plane triangle between the Earth, the centre of the planet and the position of the satellite in its orbit. The preceding formulæ given by Marth are rigorous, but generally it will be sufficient to use the first two equations of (9). If we neglect the difference between ρ and ρ', and express s and r in seconds of arc, we have

$$\left.\begin{aligned} s \sin (p-k) &= \frac{r}{\rho} . \cos (H+u) \\ s \cos (p-k) &= \frac{r}{\rho} . \sin (H+u) \cos h \end{aligned}\right\}. \qquad (10)$$

Equations (10) are convenient for use if from the observed values of s and p we wish to compute r and u. For this purpose we may use also the auxiliary quantities of Bessel, which give

$$r . \sin [\tfrac{1}{2}(F+G)+u] = \frac{s}{2} . \frac{\sin g \sin p + \sin f \cos p}{\sin f \sin g \cos \frac{1}{2}(F-G)},$$
$$r . \cos [\tfrac{1}{2}(F+G)+u] = \frac{s}{2} . \frac{\sin g \sin p - \sin f \cos p}{\sin f \sin g \sin \frac{1}{2}(F-G)};$$

but it is more convenient to use the formulæ of Marth.

In computing the quantities f, F, g, G, h, H, we use the quantities J and N, which are themselves subject to slow changes on account of precession. But we may easily form a table of the values of J and N which will

enable us to take account of these changes in the values of the auxiliary quantities. If moreover we wish to take account also of the nutation, and denote by $\Delta\omega$ and Δl the nutation in the obliquity of the ecliptic and the longitude, we may employ the differential equations of a spherical triangle.

If ω be the obliquity of the ecliptic we have

$$dJ = \cos N . \Delta\omega - \sin N \sin\omega . \Delta l,$$

$$d\mathrm{w} = \frac{\sin N . \Delta\omega + \cos N \sin\omega . \Delta l}{\sin J},$$

$$dN = -\cos J . d\mathrm{w} + \cos\omega . \Delta l,$$

or $$dN = -(\sin N . \Delta\omega + \cos N \sin\omega . \Delta l) \operatorname{cotg} J + \cos\omega . \Delta l.$$

The first and last of these formulæ will give the corrections to the values of J and N.

(2). If we assume values of the elements of the orbit of a satellite, the preceding formulæ enable us to compare our observations with the elements. In order to correct the assumed elements we must form equations of condition by differentiating the first two of equations (5). Denoting the partial differential coefficients by a, b, c, d, e, f; and by ν the residual = computed — observed value, the equation of condition will be of the form

$$a . dE + b . edP + c . de + d . d\Delta + e . \sin J\, dN + f . dJ + \nu = 0.$$

Each observation gives two equations, one from the value of ξ, and one from the value of η. The changes in the value of ζ can be neglected.

If we denote by n the longitude of the ascending node of the orbit of the satellite on the Ecliptic, by i its inclination to the Ecliptic, by w the distance of the Ecliptic from the Equator, measured on the orbit of the satellite, by v the longitude of the satellite in its orbit, we have

$$v - n + \mathrm{w} = u, \qquad \text{and } \varphi = v - P.$$

If we put

$$f' = \frac{\rho_0}{\rho}\Delta \sin f; \quad g' = \frac{\rho_0}{\rho}\Delta \sin g; \quad h' = \frac{\rho_0}{\rho}\Delta \sin h,$$

$$F' = F + \mathrm{w} - n; \quad G' = G + \mathrm{w} - n; \quad H' = H + \mathrm{w} - n,$$

the values of ξ, η and ζ become

$$\omega . \xi = r . f' \sin(F' + v),$$

$$\omega . \eta = r . g' \sin(G' + v),$$

$$\omega . \zeta = r . h' \sin(H' + v).$$

The value of $\omega . \xi$ may be written in the form

$$\omega . \xi = f' . [\sin(F' + P) . r\cos(v - P) + \cos(F' + P) . r\sin(v - P)].$$

Differentiating with respect to the elements in the plane of the orbit we have

$$\omega . d\xi = f' \sin(F' + P) . d[r\cos(v - P)] + f' \cos(F' + P) . d[r\sin(v - P)] + rf' \cos(F' + v) . dP.$$

We have also, since $v—P$ is the true anomaly,

$$r \,.\, \cos (v-P) = \cos \varepsilon - e,$$
$$r \,.\, \sin (v-P) = \sin \varepsilon \sqrt{(1-e^2)},$$
$$E-P = \varepsilon - e \sin \varepsilon,$$

and therefore

$$d \,.\, [r \cos (v-P)] = -\sin \varepsilon \,.\, d\varepsilon - de,$$
$$d \,.\, [r \sin (v-P)] = \cos \varepsilon \,.\, \sqrt{(1-e^2)} \,.\, d\varepsilon - \frac{e \sin \varepsilon\, de}{\sqrt{(1-e^2)}},$$
$$d \,.\, (E-P) = (1 - e \cos \varepsilon) \,.\, d\varepsilon - \sin \varepsilon \,.\, de,$$

or

$$d\varepsilon = \frac{d\,(E-P) + \sin \varepsilon \,.\, de}{1 - e \cos \varepsilon}.$$

Eliminating $d\varepsilon$ we have

$$d \,.\, [r \cos (v-P)] = -\frac{\sin \varepsilon}{1 - e \cos \varepsilon} \,.\, d\,(E-P) - \left(1 + \frac{\sin \varepsilon^2}{1 - e \cos \varepsilon}\right) . \, de,$$
$$d \,.\, [r \sin (v-P)] = \frac{\cos \varepsilon \,.\, \sqrt{(1-e^2)}}{1 - e \cos \varepsilon} \,.\, d(E-P)$$
$$+ \left(\frac{\cos \varepsilon \,.\, \sqrt{(1-e^2)}}{1 - e \cos \varepsilon} - \frac{e}{\sqrt{(1-e^2)}}\right) . \sin \varepsilon \; de.$$

But since

$$\frac{\sin \varepsilon}{1 - e \cos \varepsilon} = \frac{\sin (v-P)}{\sqrt{(1-e^2)}} : \text{ and } \cos (v-P) = \frac{\cos \varepsilon - e}{1 - e \cos \varepsilon},$$

the values of these differentials become

$$d \,.\, [r \cos (v-P)] = -\frac{\sin (v-P)}{\sqrt{(1-e^2)}} \,.\, d\,(E-P) - \left(1 + \frac{\sin \varepsilon \sin (v-P)}{\sqrt{(1-e^2)}}\right) . de,$$
$$d \,.\, [r \sin (v-P)] = \frac{\cos (v-P) + e}{\sqrt{(1-e^2)}} \,.\, d\,(E-P) + \frac{\sin \varepsilon \cos (v-P)}{\sqrt{(1-e^2)}} . \, de.$$

Multiplying the first differential by $f' \sin (F'+P)$, and the second by $f' \cos (F'+P)$, the value of $\omega \,.\, d\xi$ becomes

$$\omega \,.\, d\xi = \frac{f'}{\sqrt{(1-e^2)}} . \left\{ \cos (F'+v) + e \cos (F'+P) \right\} . \, dE$$
$$+ \frac{f'}{\sqrt{(1-e^2)}} . \left\{ \cos (F'+v) [r\sqrt{(1-e^2)} - 1] - e \cos (F'+P) \right\} . \, dP$$
$$+ \frac{f'}{\sqrt{(1-e^2)}} . \left\{ \cos (F'+v) \sin \varepsilon - \sin (F'+P) \,.\, \sqrt{(1-e^2)} \right\} . \, de.$$

If we substitute the value of $r = 1 - e \cos \varepsilon$, in the coefficient of dP, and observe that the value of the partial differential coefficient d is evident, we shall have the values of the first four coefficients given by Bessel; Astronomische Nachrichten, Bd. 9, p. 10.

In order to find the two other coefficients, we take the value of $\omega . \xi$

$$\omega . \xi = r . \frac{\rho_0}{\rho} \Delta . [\sin u \cos(a—N) \cos J—\cos u \sin(a—N)].$$

Differentiating with respect to the elements that determine the position of the plane of the orbit we have

$$\omega . d\xi = r . \frac{\rho_0}{\rho} \Delta . \Big\{ [\cos u \cos(a—N) \cos J + \sin u \sin(a—N)] . du$$
$$+ [\sin u \sin(a—N) \cos J + \cos u \cos(a—N)] dN — \sin u \cos(a—N) \sin J dJ \Big\}$$

The values of the auxiliary quantities f and F give

$$\omega . d\xi = r . \frac{\rho_0}{\rho} \Delta . \Big\{ \sin f \cos(F'+v) . du + \sin f \cos(F'+v) \cos J . dN$$
$$— \cos u \cos f \sin J . dN + \sin u \cos f . dJ \Big\}.$$

If we now consider the spherical triangle formed by the equator, the ecliptic and the orbit of the satellite, we shall have by the formulæ for the differentials of a spherical triangle,

$$d\mathrm{w} = \cos i . dn — \cos J . dN,$$

$$dn = —\frac{\sin \mathrm{w}}{\sin i} . dJ + \frac{\cos \mathrm{w} \sin J}{\sin i} . dN,$$

or

$$d\mathrm{w} = —\frac{\sin \mathrm{w} \cos i}{\sin i} . dJ + \frac{\cos \mathrm{w} \cos i \sin J}{\sin i} . dN — \cos J . dN.$$

We have therefore

$$d(\mathrm{w}—n) = (—\cos J — \cos \mathrm{w} \sin J \operatorname{tang} \tfrac{1}{2}i) . dN + \sin \mathrm{w} \operatorname{tang} \tfrac{1}{2}i . dJ.$$

Since $du = d(\mathrm{w}—n)$, by substituting this value of du we have

$$\omega . d\xi = —\sin J . \Big\{ f' . \operatorname{tang} \tfrac{1}{2}i \cos \mathrm{w} . r \cos(F'+v) + \frac{\rho_0}{\rho} \Delta \cos f . r \cos u \Big\} . dN$$
$$+ \Big\{ f' . \operatorname{tang} \tfrac{1}{2}i \sin \mathrm{w} . r \cos(F'+v) + \frac{\rho_0}{\rho} \Delta \cos f . r \sin u \Big\} . dJ.$$

The values therefore of the partial differential coefficients a, b, c, d, e, f, are as follows:

$$a = \omega . \frac{d\xi}{dE} = \frac{f'}{\sqrt{(1—e^2)}} . \Big\{ \cos(F'+v) + e \cos(F'+P) \Big\}$$

$$b = \omega . \frac{d\xi}{e dP} = \frac{—f'}{\sqrt{(1—e^2)}} . \Big\{ \cos\Big(F'+v\Big) . \Big(\cos \varepsilon \sqrt{(1—e^2)}$$
$$+ \frac{e}{1+\sqrt{(1—e^2)}}\Big) + \cos(F'+P) \Big\}$$

$$c = \omega . \frac{d\xi}{de} = \frac{f'}{\sqrt{(1—e^2)}} . \Big\{ \cos(F'+v) \sin \varepsilon — \sin(F'+P) . \sqrt{(1—e^2)} \Big\}$$

$$d = \omega . \frac{d\xi}{d\varDelta} = \frac{\omega . \xi}{\varDelta},$$

$$e = \omega . \frac{d\xi}{\sin J dN} = -f' . \operatorname{tang} \tfrac{1}{2} i \cos \mathrm{w} . r \cos (F' + v) - \frac{\rho_0}{\rho} \varDelta . \cos f . r \cos u$$

$$f = \omega . \frac{d\xi}{dJ} = f' . \operatorname{tang} \tfrac{1}{2} i \sin \mathrm{w} . r \cos (F' + v) + \frac{\rho_0}{\rho} \varDelta . \cos f . r \sin u. \quad (11)$$

In order to obtain the coefficients in η we have only to put g' and G' in the place of f' and F'.

The preceding formulæ give the values of the differential coefficients adapted to the quantites

$$\xi = x = s \sin p : \qquad \eta = y = s \cos p;$$

and we must therefore combine our observed quantities s and p to find the residuals for the equations of condition. Generally it is better to compare directly with each of the observed quantities, since one of them may be erroneous and the other not. If we wish to do this in the present case, we must use the formulæ

$$ds = \sin p . dx + \cos p . dy,$$
$$s . dp = \cos p . dx - \sin p . dy.$$

By means of equations (11) the value of dx is

$$dx = a . dE + b . edP + c . de + d . d\varDelta + e . \sin J dN + f . dJ,$$

with a similar value for dy. Substituting these values for dx and dy in the preceding equations we shall have the equations of condition for ds and sdp. If we make

$$m \sin M = f' \cos (F' + v), \qquad n' \sin N' = f' \cos (F' + P),$$
$$m \cos M = g' \cos (G' + v), \qquad n' \cos N' = g' \cos (G' + P);$$

$$l \sin L = f' \sin (F' + P), \qquad k' \sin K' = \frac{\rho_0}{\rho} \varDelta \cos f,$$

$$l \cos L = g' \sin (G' + P), \qquad k' \cos K' = \frac{\rho_0}{\rho} \varDelta \cos g.$$

The values of the differential coefficients for ds and sdp are as follows, where $\sin \varphi'$ is put for e on the right-hand side of the equations.

For the distances;

$$\frac{ds}{dE} = \frac{1}{\cos \varphi'} . \left\{ m \cos (M - p) + n' \cos (N' - p) \sin \varphi' \right\}$$

$$\frac{ds}{edP} = \frac{-1}{\cos \varphi'} . \left\{ m \cos (M - p) (\cos \varepsilon \cos \varphi' + \operatorname{tang} \tfrac{1}{2} \varphi') + n' \cos (N' - p) \right\}$$

$$\frac{ds}{de} = \frac{1}{\cos \varphi'} . \left\{ m \cos (M - p) \sin \varepsilon - l \cos (L - p) \cos \varphi' \right\}$$

$$\frac{ds}{d\varDelta} = \frac{s}{\varDelta}$$

$$\frac{ds}{\sin JdN} = r.\left\{ m \cos(M-p) \cos \mathrm{w} \operatorname{tang} \tfrac{1}{2}i + k' \cos(K'-p) \cos u \right\}$$

$$\frac{ds}{dJ} = -r.\left\{ m \cos(M-p) \sin \mathrm{w} \operatorname{tang} \tfrac{1}{2}i + k' \cos(K'-p) \sin u \right\}. \tag{12}$$

For the angles:

$$\frac{s\,dp}{dE} = \frac{1}{\cos\varphi'}.\left\{ m \sin(M-p) + n' \sin(N'-p) \sin\varphi' \right\}$$

$$\frac{s\,dp}{edP} = \frac{-1}{\cos\varphi'}.\left\{ m \sin(M-p)(\cos\varepsilon\cos\varphi' + \operatorname{tang}\tfrac{1}{2}\varphi') + n' \sin(N'-p) \right\}$$

$$\frac{s\,dp}{de} = \frac{1}{\cos\varphi'}.\left\{ m \sin(M-p) \sin\varepsilon - l \sin(L-p) \cos\varphi' \right\}$$

$$\frac{s\,dp}{d\varDelta} = 0$$

$$\frac{s\,dp}{\sin JdN} = -r.\left\{ m \sin(M-p) \cos \mathrm{w} \operatorname{tang} \tfrac{1}{2}i + k' \sin(K'-p) \cos u \right\}$$

$$\frac{s\,dp}{dJ} = r.\left\{ m \sin(M-p) \sin \mathrm{w} \operatorname{tang} \tfrac{1}{2}i + k' \sin(K'-p) \sin u \right\} \tag{13}$$

Generally in equations (12) and (13) we may put $\varphi' = 0$, and these equations will become simpler.

The values of i, n and w will be found from the spherical triangle between the Equator, the Ecliptic and the orbit of the satellite. This triangle gives by means of the Gaussian equations, if we denote by ω the obliquity of the ecliptic;

$$\left.\begin{aligned} \cos\tfrac{1}{2}i \sin\tfrac{1}{2}(\mathrm{w}+n) &= -\cos\tfrac{1}{2}N \cos\tfrac{1}{2}(\omega-J) \\ \cos\tfrac{1}{2}i \cos\tfrac{1}{2}(\mathrm{w}+n) &= \sin\tfrac{1}{2}N \cos\tfrac{1}{2}(\omega+J) \\ \sin\tfrac{1}{2}i \sin\tfrac{1}{2}(\mathrm{w}-n) &= \cos\tfrac{1}{2}N \sin\tfrac{1}{2}(\omega-J) \\ \sin\tfrac{1}{2}i \cos\tfrac{1}{2}(\mathrm{w}-n) &= -\sin\tfrac{1}{2}N \sin\tfrac{1}{2}(\omega+J) \end{aligned}\right\}. \tag{14}$$

(3) In what precedes it is assumed that the elements of the orbit of the satellite are known, and the formulæ given are sufficient for comparing the observed positions of the satellite with those computed from the elements, and for forming the equations of condition necessary to correct the assumed values of these elements. In order to obtain an approximate knowledge of the orbit of a satellite various methods may be used, but in this case, the elaborate formulæ given for computing the orbit of a planet are not necessary; and in my own experience I have found it nearly useless to employ

them. The reason of this is that the small errors of observation exert too great an influence on the resulting elements, so that it is better to use simpler methods. If we assume the orbit of the satellite to be circular, and determine by observation the form of the apparent ellipse described by the satellite, and the position of its major axis, the solution of a spherical triangle will give the values of N and J, the quantities which fix the position of the plane of the orbit. With these quantities we can compute the values of the auxiliaries f, g, h, &c., and by means of equations (10), and the observed values of s and p, we shall have the values of r and u, the polar coordinates of the satellite in the plane of its orbit. It is then easy to determine the elements of the ellipse described by the satellite around its primary, the position of the line of apsides, and the place of the satellite in its orbit at a given time.

The following figure explains the geometrical meaning of the auxiliary quantities; and one will be able to deduce the formulæ for computing them directly from the spherical triangles.

$TZ = f$,

$TY = g$,

$TX = h$,

$OTZ = F$,

$OTY = G$,

$OTX = H$;

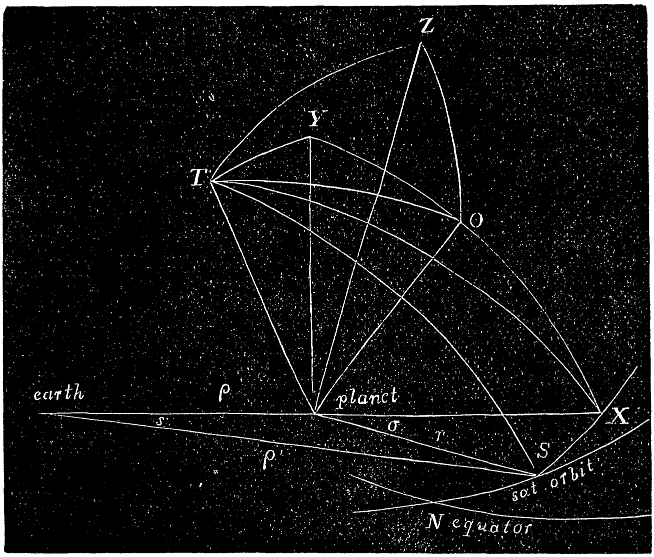
$OT = J =$ inclination of orbit to Equator,

$OY = \delta$,

$OX = 90° - \delta$,

P, the planet,

S, the satellite.



$$STX = 90° - (H + u),$$
$$TXO = 180° - k,$$
$$TXS = 180° - (p - k),$$
$$TOZ = 270° - (a - N).$$

I am indebted to Professor Frisby for verifying my derivation of the formulæ (12) and (13), which will be found, I hope, correct, and in a form convenient for use.